An approach to Pythagoras' Theorem

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Abstract:

There are over 400 proofs, algebraic, graphic, animated of Pythagoras' Theorem. But how, particularly, did *his* proof, or the proof named after him, come about? Nobody knows.

All proofs start with a right-angled triangle, necessarily so, including Euclid's.

However, we use a quite different approach. It is more in the nature of an exploration - an exploration of the applicability of similar triangles. This leads us to the notion and definition of a *right angle* and a *right-angled triangle*, finally resulting, in a sense, a 'discovery' of Pythagoras' Theorem.

Starting at a fundamental level, with an arbitrary, (non-isosceles), triangle and using elementary geometry of only simple proportions or ratios of similar triangles, we use a construction which necessitates considering the triangle to be right-angled, then leading easily on to the Theorem.

1. Similar Triangles

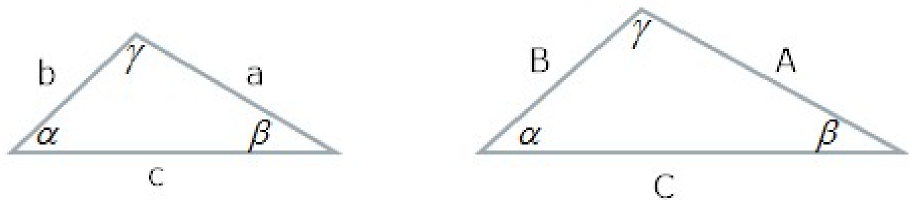
First we require that we can agree that we can decide whether one angle is greater than another (and hence also if they are equal).

Second, if a line is drawn and another line cuts it and the angles on each side of the cut are equal we call each angle a right angle ([1, Book I, Definition 10]). This is a purely geometrical definition without any notion of a numerical measure or value given to an angle, so that we do not state that a right angle is 90°. A triangle with one angle which is a right angle is a right-angled triangle ([1, Book I, Definition 21]). We also do not state that the sum of the angles of a triangle is 180°. Although we have given a definition of a right angle here, our development **forces** the use of a right angle in its construction.

So let us embark on this exploration of applicability of similar triangles.

The two triangles shown are similar and their corresponding angles are equal. The sides of either triangle are in the same proportion to the corresponding sides of the other.

Figure 1 Similar Triangles



The angles α, β, γ are the same in both triangles.

From this we have

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}.$$

For example

$$\frac{1}{2} = \frac{3}{6} = \frac{4}{8}$$

But this can be written

$$\frac{a}{b} = \frac{A}{B} \qquad \frac{a}{c} = \frac{A}{C} \qquad \frac{b}{c} = \frac{B}{C}.$$

Thus for the numerical example above

$$\frac{1}{3} = \frac{2}{6} \qquad \frac{1}{4} = \frac{2}{8} \qquad \frac{3}{4} = \frac{6}{8}$$

Now this involves proportions, or ratios on the left-hand sides of the equations which refer to the first triangle whilst ratios on the right-hand sides refer to the second triangle.

What can we do with this?

Well if we could divide the triangle into two similar sub-triangles, then only the sides of the original triangle would be involved in ratios - and we might expect that we could then get some relation between the sides of this original triangle.

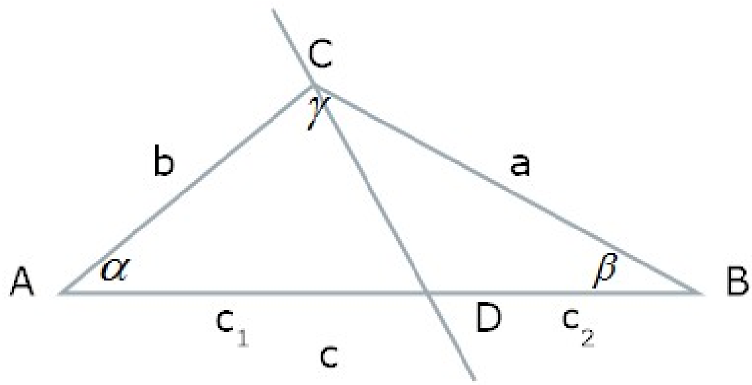
The creative thought!

2. The Construction

With a non-isosceles triangle one angle, say γ, is greater than one or both of the other two.

So let us draw a line through the vertex C of the triangle ABC at this greatest angle – as shown, whose sides CA and CB contain γ. (If two angles are equal and greatest any one can be chosen.) Let the line cut AB at D and let $AD = c_1$ and $DB = c_2$ so that $c = c_1 + c_2$.

Figure 2 A Divided Triangle



$\triangle BCD : \dfrac{c_2}{a} = \dfrac{a}{c} : \triangle ABC$ $\qquad$ $\triangle ADC : \dfrac{c_1}{b} = \dfrac{b}{c} : \triangle ABC$

So we have two triangles within the original triangle ABC.

3. The First similar triangle within the original triangle

The line CD is quite arbitrary so we want to make it so that, say triangle BCD, is similar to the original triangle ABC. This can always be done.

Simply draw CD so that the angle $\widehat{BCD}$ is α (it doesn't look it).

You may ask − why not make it β? Because we would be making an isosceles triangle BCD which would not necessarily be similar to triangle ABC.

Then for $\triangle BCD$ to be similar to $\triangle ABC$ we must have $\widehat{BDC}$ to be γ.

We can now write the ratios

$$\triangle BCD : \frac{c_2}{a} = \frac{a}{c} : \triangle ABC$$

or

$$c_2 = \frac{a^2}{c}. \tag{1}$$

4. The Second similar triangle within the original triangle

If now we want triangle ACD to also be similar to the original triangle ABC we should expect some constraints on the type of triangle ABC is in the first place.

Now $\widehat{ACB}$ is γ and we cannot also have $\widehat{ACD}$ as γ as well ($\alpha \neq 0$). Hence $\widehat{ACD}$ must be β and then $\widehat{ADC}$ is γ.

So we have $\widehat{ADC} = \gamma = \widehat{BDC}$. That is, these angles are right angles. Then too, as $\widehat{ACB} = \gamma$ it also must be a right angle. This is the constraint on $\triangle ABC$ − that is, that it should be a right-angled triangle!

This now being the case, with $\triangle ADC$ similar to $\triangle ABC$, we have the ratios

$$\triangle ADC : \frac{c_1}{b} = \frac{b}{c} : \triangle ABC$$

or

$$c_1 = \frac{b^2}{c}. \tag{2}$$

5. Pythagoras' Theorem

Adding equations (1) and (2) we get

$$c = c_1 + c_2 = \frac{a^2 + b^2}{c}.$$

or

$$c^2 = a^2 + b^2.$$

To the author's knowledge this derivation is original.

6. History and Speculation

Dates for Pythagoras are c.570 - c.495 BC whilst dates for Euclid are c.330 - c.275BC, some 200 years after Pythagoras. Euclid's *Elements* contains the results of many previous writers, mathematicians and logicians. His *Elements,* Book I, Proposition 47, contains Pythagoras' Theorem stating "*In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.*" (There is no mention of an 'hypotenuse'.) Clearly from this, the notion of a right angle and a right-angled triangle were known. Euclid's proof of Pythagoras' Theorem is universal.

The Theorem was known over 1,000 years before Pythagoras, by the Babylonians, but no 'proof' is known. The Chinese and Indians were also aware of it.

All proofs of the Theorem, necessarily start off with a right-angled triangle. Assuming a right-angled triangle (unlike here), the proof by similar triangles has been well-known since early Greek times. A generalization of Pythagoras is the well-known trigonometric *cosine* formula.

The development and use of similarity occurs much later in Euclid's *Elements,* Book VI, and so it is considered by some that the notion of similarity was not employed by Pythagoras. And it is probable that early Greeks thought more in terms of squares as an area rather than proportions - Euclid's proof uses that notion.

However, we have used similar triangles and have been at pains to explore its applicability to an *arbitrary* triangle (at first) following where it leads us and we find that we need the notion of a right angle (as originally defined in the *Elements*) and a right-angled triangle.

It is therefore of some wonder whether Pythagoras (or Pythagoreans) also played around with this idea with similarity in mind, seeing where it led him (them). Since the Pythagoreans were a secretive group, it may well have been something they never revealed!

So may we just remark: Pythagoras - he did it (t)his way?!

References

[1] Euclid's Elements,
http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html.